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Some Properties of Topological Geons

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We investigate the Finkelstein–Misner geons for a non-simply-connected spacetime manifold (M, g_0). We use relations between different Lorentzian structures unequivalent to g_0 and topological properties of M given by the Morse theory. It implies that to some pieces of geons we have to associate Wheeler's "wormholes." Geons that correspond to time-orientable Lorentz structures are related to g_0 by Morse functions that describe the attaching of a handle of index one. In the case of geons associated to time-nonorientable Lorentzian structures, appropriate handles are related to loops along which the notion of time reverses. If we assume electromagnetic properties of geons, then only four species, "ν", "e", "p", "m", of different geons can exist and geon "m" has to decay according to "m"$\rightarrow$"ν"+"p"+"e".

1. INTRODUCTION

Let us assume that the background of any physical theory is given by some differential 4-manifold $\mathcal{M}$. Further let us assume that $\mathcal{M}$ is a connected, open, orientable spin manifold whose fundamental group $\pi_1(\mathcal{M})$ has an infinite, cyclic subgroup. Now we should add a metric g to make a differential manifold $\mathcal{M}$ a Lorentzian manifold. Usually we assume that it is done in such a way that $(\mathcal{M}, g)$ is a causal (i.e., it contains no close timelike or null curves) and isochronous (i.e., a continuous choice of the forward light cone can be made) manifold. There is a common view that if there exists a close timelike curve along which the notion of time reverses, then no physical entity could survive a trip along such a curve.

Let us assume that there exist objects which are related to different Lorentzian metrics on $\mathcal{M}$. Such pure "gravitational structures" have been introduced by Finkelstein and Misner (1959) and were called by them M-geons. According to these authors any M-geon is given by a homotopy class of Lorentzian structures of $\mathcal{M}$. They have considered the case when

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the compactification of a spacelike hypersurface is topologically S^3 and when each M-geon is given by a homotopy class of maps from S^3 to RP^3. In this case RP^3 is the deformation retract of the space of all Lorentzian structures of a tangent space at a point $x \in \mathcal{M}$.

In this paper the notion of the M-geon is the fundamental one; however, we will use the following, slightly more general definition.

Definition 1. Let $\mathcal{M}$ be an open 4-manifold satisfying the condition mentioned at the beginning of this introduction. Let us fix some concrete Lorentzian metric g_σ in any homotopy class σ of Lorentzian structures on $\mathcal{M}$ (not necessarily time-orientable). Now a concrete Finkelstein–Misner–Wheeler geon (FMW-geon for brevity) is determined by a metric structure on g_σ, uniquely given by its homotopy class σ.

The reason we add Wheeler's name to this definition will be clear later.

The set of homotopy classes of Lorentzian structures on an open 4-manifold $\mathcal{M}$ is given by (Bugajska, 1987a)

$$H^1(\mathcal{M}, Z_2) \times H_1^\infty(\mathcal{M}, Z) \tag{1}$$

So we have that the number and the nature of FMW-geons depend on the algebraic-topological invariants of $\mathcal{M}$.

A manifold $\mathcal{M}$ is characterized by its fundamental group $\pi_1\mathcal{M}$, its homotopy, homology, and cohomology groups, its orientability properties, characteristic classes, etc. Most of these have some physical meaning. Some of them (for example, the Stiefel–Whitney classes) do not depend on a concrete differential structure of $\mathcal{M}$ and are topological invariants of the manifold $\mathcal{M}$ (Steenrod, 1951). Nevertheless, we assume that $\mathcal{M}$ is equipped with some concrete differential structure. It is known that any such manifold $\mathcal{M}$ possesses a smooth triangulation making it homeomorphic to a combinatorial manifold (Hilton, 1968). (By the Whitehead theorem every differential manifold carries an essentially unique smooth combinatorial structure.) Let W be a Whitehead functor from the category of differential manifolds Diff to the category of combinatorial manifolds PL and let F be the forgetful functor, i.e.,

$$\mathrm{Diff} \xrightarrow{W} PL \xrightarrow{F} \mathrm{Top}$$

It appears that there exist combinatorial manifolds that do not admit any differential structure (Hilton, 1968). In other words, for some combinatorial manifold N there does not exist a differential manifold $\mathcal{M}$ such that $F \circ W(\mathcal{M}) \cong F(N)$. Thus, if it will appear that the topological and combinatorial invariants are of primary importance for physical theories, then perhaps we will be forced to resign from the differential structure of $\mathcal{M}$. But now let us agree that the arena of our physical world is given by some differentiable manifold $\mathcal{M}$ possessing some concrete topological invariants. Although these invariants do not depend on a concrete metric structure of $\mathcal{M}$, we have some relations between them and metric characteristics, such as scalar curvature, mean curvature, or sectional curvature of possible metric structures of $\mathcal{M}$. These relations are different for Riemannian and semi-Riemannian cases. So, for example, the scalar curvature of a possible Riemannian structure seems to be unrelated to the fundamental group of $\mathcal{M}$ (Kazdan and Warner, 1975), whereas in a Lorentzian case there exist some relations [a relativistic spherical space form has to have a finite fundamental group and to be noncompact (Calabi and Marcus, 1962), etc.]. Besides, an open manifold $\mathcal{M}$ cannot admit a complete Riemannian structure whose sectional curvature is bounded below (Milnor, 1963). Moreover, using the Morse theory of geodesic and the notion of conjugated points we can obtain some other relations between the topology and curvature for Riemannian as well as for Lorentzian structures on $\mathcal{M}$ (Uhlenbeck, 1975). The assumption about the fundamental group $\pi_1\mathcal{M}$ made at the beginning of this section can be satisfied, for example, by a manifold $\mathcal{M}$ that admits a complete Riemannian structure whose sectional curvature is nonpositive. In this case $\pi_1\mathcal{M}$ has no other elements of finite order than the identity and all higher homotopy groups $\pi_i M$, $i > 1$, vanish (Milnor, 1963).

The properties of the manifold $\mathcal{M}$ described by its fundamental group $\pi_1 M$ are very important in our considerations. Namely, we have the natural homomorphism

$$h: \pi_1\mathcal{M} \to H_1(\mathcal{M}, z) \hookrightarrow H_1^{\infty}(\mathcal{M}, z)$$

This fact together with the formula (1) implies certain relations between some of the FMW-geons. In Section 2 we will see that geons related to time-orientable Lorentzian structures that are nonhomotopic to the exceptional, fundamental, and observed metric g_0 have to be associated with the attaching of a handle of index 1 in some place of our space-time manifold $\mathcal{M}$. More exactly, they are related to the g_0 structure by a Morse function which describes the mentioned handle. Equivalently, instead of a handle of index 1 we can talk about a surgery of type $(1, 3)$ or Wheeler's "wormhole." Geons related to time-nonorientable Lorentzian structures are also associated to a corresponding handle of index 1, but in a different way. Namely, these pieces of geons are associated to close curves along which the notion of time reverses. The homotopy classes of these curves are related to appropriate handles (or Wheeler's "wormholes").

Because of such a strong relation between Finkelstein–Misner M-geons and handles of index 1, we propose to denote M-geons by FMW-geons.

In Section 3 we assume that FMW-geons can possess some electromagnetic properties. This fact allows us to introduce only four qualitatively different species of FMW-geons. We denote them by "ν," "e," "p," and "m." Moreover, in this approach we obtain the following relation:

$$\text{“}m\text{''} \to \text{“}p\text{''} + \text{“}e\text{''} + \text{“}\nu\text{''}$$

2. PURE GRAVITATIONAL STRUCTURES

It is known (Hawking and Ellis, 1974) that any Lorentzian structure g of $\mathcal{M}$ can be given by some Riemannian metric $\tilde{g}$ and a tangent line bundle $L \subset T\mathcal{M}$ over $\mathcal{M}$. Of course, such "representation" of g is not unique. However, when we consider a family η of Riemannian structures related to some auxiliary splitting

$$T\mathcal{M} = X \oplus N$$

of the tangent bundle $T\mathcal{M}$ into a line bundle X and its linear complement N, then, according to Bugajska (1987a), any Lorentzian metric g is determined by a unique couple

$$g \equiv (\tilde{g}, L) \tag{2}$$

with $\tilde{g} \in \eta$. Since all Riemannian metrics belonging to η are homotopic, to find the homotopy classes of Lorentzian metrics, we have to find the homotopy classes of tangent line bundles.

We will assume the following property of our geons. Namely, every gravitational structure related to any FMW-geon can be given by the same Riemannian structure, say $\tilde{g}$, but different tangent line bundles. Let us denote any geon by $\mathcal{W}_i$. We can write

$$\mathcal{W}_i \sim (\tilde{g}, L_i) \tag{3}$$

In other words, we can say that any FMW-geon $\mathcal{W}_i$ can be related to a tangent line bundle L_i. Besides, according to our assumptions, different FMW-geons are related to different line bundles, which belong to different homotopy classes. So, to classify FMW-geons we have to know the classes of tangent line bundles over $\mathcal{M}$.

Let ξ be some line bundle over $\mathcal{M}$. It is known that the set of isomorphic classes of line bundles over $\mathcal{M}$ can be given by $H^1(\mathcal{M}, Z_2)$ (Husemoller, 1966). It can be shown that for an open manifold there is no obstruction to embed any line bundle ξ into the tangent bundle $T\mathcal{M}$ (Koschorke, 1974). In other words, for any element $\tau \in H^1(\mathcal{M}, Z_2)$ there does exist a tangent line bundle $L \hookrightarrow T\mathcal{M}$ such that its first Stiefel-Whitney class satisfies

$$w_1(L) = \tau \tag{4}$$

Moreover, it can be shown (Bugajska, 1987a) that for any line bundle ξ over $\mathcal{M}$ the set of nonhomotopic embeddings into $T\mathcal{M}$ is given by $H_1^\infty(\mathcal{M}, Z)$. In other words, for any $\tau \in H^1(\mathcal{M}, Z_2)$ we have the set given by $H_1^\infty(\mathcal{M}, Z)$ of homotopy classes of tangent line bundles. So, in principle there could exist $H^1(\mathcal{M}, Z_2) \times H_1^\infty(\mathcal{M}, Z)$ different FMW-geons.

Now it is natural to assume, and we always do this, that all observers and all observations are related to the same concrete Lorentz structure g_0 on $\mathcal{M}$. This means, among other things, that any description of observable physical objects is given by sections of an appropriate vector bundle associated to the principal spin bundle of the g_0 structure. In other words, $(\mathcal{M}, g_0)$ is our space-time manifold. However, the causal and the isochronous assumptions mentioned in the introduction are too weak to be very useful. So usually we assume that our space-time manifold $(\mathcal{M}, g_0)$ is strongly causal and Lorentz-complete. [The former condition says that every point of $\mathcal{M}$ is contained in an open neighborhood that intersects every timelike or null curve in a connected set. The latter condition means that for each pair of points $x, y \in \mathcal{M}$ the set $I^+(x) \cap I^-(y)$ is compact; here $I^-(x)$ and $I^+(x)$ denote the past or the future of x, respectively (Hawking and Ellis, 1974).] It was shown by Choquet-Bruhat (1967) and Lichnerowicz (1967) that strongly causal plus Lorentz-complete is equivalent to globally hyperbolic. However, due to the Geroch result, for any globally hyperbolic Lorentzian manifold $(\mathcal{M}, g_0)$ there exists a splitting

$$\mathcal{M} = S \times R \tag{5}$$

such that the surfaces (S, t), $t \in R$, are all Cauchy surfaces. Usually such a splitting (5) is called a globally hyperbolic splitting. Thus, the Finkelstein-Misner M-geons correspond to the case when the compactification of S is given by S^3. In this case (as well as in the case when a Cauchy surface in a globally hyperbolic splitting is S^3 itself) the group $H^1(\mathcal{M}, Z_2) = 0$ and all FMW-geons correspond to time-orientable metric structures. However, we will admit that our manifold $\mathcal{M}$ is topologically more complicated and that the fundamental group $\pi_1\mathcal{M}$ is nontrivial and contains infinite cyclic subgroups.

Let us recall the known natural relation between the integral first homology group of $\mathcal{M}$ and the fundamental group of $\mathcal{M}$. Namely we have the theorem (Hu, 1959) that if $\mathcal{M}$ is pathwise connected, then the natural homomorphism

$$h\colon \pi_1(\mathcal{M}, x) \to H_1(\mathcal{M}, Z) \tag{6}$$

has the commutator subgroup of $\pi_1(\mathcal{M}, x)$ as its kernel. This means that we have the natural isomorphism h^* between the group $H_1(\mathcal{M}, Z)$ and $\pi_1\mathcal{M}$ made abelian. But we have $H_1(\mathcal{M}, Z) \hookrightarrow H_1^\infty(\mathcal{M}, Z)$ and this latter group numerates the homotopy classes of isomorphic tangent line bundles over $\mathcal{M}$. So we see immediately that for such homotopically nontrivial manifold $\mathcal{M}$ we should have certain relations between some FMW-geons and some classes of homotopic loops in $\mathcal{M}$. To find these relations, let us consider a realization of $\mathcal{M}$ by an expanding union of compact manifolds $\{U_i\}$ with boundaries. Let

$$\mathcal{M} = \bigcup_{i=0}^{\infty} U_i \tag{7}$$

and $U_i \subset U_{i+1}$, with U_0 being a 4-cell. Now either U_{i+1} is a collarlike neighborhood of U_i or U_{i+1} is U_i with a handle of index $\lambda \leqq n-1$ attached. In the latter case we can say that ∂U_{i+1} can be obtained from ∂U_i by a surgery of type $(\lambda, n-1)$. Let $U_i \subset \mathcal{M}$ be such that ∂U_{i+1} is obtained from ∂U_i by a surgery of type $(1, 3)$ (or equivalently that we can get U_{i+1} by attaching a handle of index 1 to U_i). In other words, we can say that ∂U_{i+1} and ∂U_i are related by a spherical modification (Milnor, 1963) of type $\lambda - 1 = 0$. Let us notice that it could correspond to the formation of a Wheeler "wormhole",

Now, let ρ be a loop that represents the homotopy class of loops given by such an effective attaching of a handle of index 1 to U_i (Bugajska, 1987a). Let $\sigma \in H_1(\mathcal{M}, Z)$ be a generator of the torsionless part of $H_1(\mathcal{M}, Z)$, which corresponds to the homotopy class of ρ under the isomorphism h^*. Let L_0 be the trivial tangent line bundle that corresponds to our space-time manifold $(\mathcal{M}, g_0)$ [i.e., $g_0 \equiv (\tilde{g}, L_0)$] and let L_1 be another trivial tangent line bundle, which belongs to the homotopy class determined by the element $\sigma \in H_1(\mathcal{M}, Z)$ mentioned above. It can be seen (Bugajska, 1987a) that two Lorentzian structures $g_0 \equiv (\tilde{g}, L_0)$ and $g_1 \equiv (\tilde{g}, L_1)$ are related by a Morse function f on $\mathcal{M}$ with only one nondegenerate critical point of index 1. This function describes exactly the attaching of a handle of index 1 to U_i in the decomposition (7) of $\mathcal{M}$. So if FMW-geons $\mathcal{W}_0 \sim L_0$ and $\mathcal{W}_1 \sim L_1$ do exist, then the main difference between them is that $\mathcal{W}_1$, in contrast to $\mathcal{W}_0$, has to distinguish one Wheeler "wormhole" in our space-time manifold $\mathcal{M}$. In other words, we can say that a FMW-geon $\mathcal{W}_0$ is not "sensitive" to any topological properties of $\mathcal{M}$, whereas $\mathcal{W}_1$ has to recognize the attaching handle of index 1 in some decomposition of $\mathcal{M}$ of the form (7). Hence, any generator of $H_1(\mathcal{M}, Z)$ that corresponds to the homotopy class of loops determined by the attaching of a handle of index 1 (or equivalently to the Wheeler "wormhole") can be associated to a FMW-geon in a natural way. Now it is obvious why we call such pure gravitational structures $\mathcal{W}_i$ FMW-gons.

However, we have two possible physical interpretations of such structures. Namely, we can assume that FMW-geons that "recognize" different Wheeler "wormholes" (or equivalently that are related to different handles of index 1) correspond to (a) qualitatively different objects or (b) different concrete objects of the same quality.

Possibility (a) seems to involve greater difficulties, requiring that an FMW-geon that is related to one part of the universe, namely the part that surrounds a corresponding handle of index 1 (or equivalently one Wheeler "wormhole"), is qualitatively different from a similar FMW-geon related to another part of the universe. This suggests that physics may be different in different parts of the space-time manifold $(\mathcal{M}, g_0)$ or that different parts of the universe have different natures. This would contradict our understanding of physical principles. So it seems that possibility (a) should be rejected.

Possibility (b) seems to be in better agreement with our intuition. It simply means that all FMW-geon corresponding to $\sigma \in H_1(\mathcal{M}, Z)$ in the way described above are the same kind of physical "beings." The characteristic feature of these "beings" is that they are related to the fundamental Lorentzian structure g_0 [which determines our observable space-time $(\mathcal{M}, g_0)$] by a Morse function that describes a handle of index 1 in some place of our space-time manifold $\mathcal{M}$. Intuitively speaking, these objects have to "recognize" one Wheeler "wormhole."

However, besides the possibilities considered above, we should also decide whether FMW-geons are (1) "localized" objects or (2) global objects. In case 1 we have still at least two possibilities:

(1i) "Localizations" of a given FMW-geon $\mathcal{W}_i$ are completely unrelated to "positions" of the corresponding handle. This means that a geon $\mathcal{W}_i$ "observes" almost everywhere a metric structure of $\mathcal{M}$ given by g_0. Only when it is close to an appropriate handle does it start to "observe" it, i.e., it starts to observe different metric relations.

(1ii) Any FMW-geon $\mathcal{W}_i$ is always situated in the closest vicinity of the corresponding "wormhole." So any movement of $\mathcal{W}_i$ is accompanied by the corresponding movement of the appropriate handle of index 1 and vice versa.

Possibility 2 says that any FMW-geon $\mathcal{W}_i$ is smeared over the whole universe $\mathcal{M}$. In this case the interpretation (a) given above could be even more acceptable than (b). It seems, however, that the combination (b) + (1ii) is the closest one to the hypothesis of Finkelstein, Misner, and Wheeler as well as to our intuition.

3. FMW-GEONS WITH ELECTROMAGNETIC PROPERTIES

By definition, FMW-geons $\mathscr{W}_i$ are objects related to certain non-homotopic Lorentzian structures g_i on $\mathscr{M}$. Since, according to our assumption, any g_i is determined by a couple $(\tilde{g}, L_i)$ with the same Riemannian metric $\tilde{g}$, any geon $\mathscr{W}_i$ is characterized by an appropriate tangent line bundle L_i. However, although the homotopy classes of tangent line bundles are given by the formula (1), we have no natural correspondence between these homotopy classes and elements of (1). To find any such correspondence, we have to fix some line bundle and relate it to the trivial element of $H_1^\infty(\mathscr{M}, Z)$. We have to do this in any class of isomorphic tangent line bundles over $\mathscr{M}$. So we see that the existence of the exceptional Lorentzian structure g_0 (related to all observations and all observers) is very important. It simply fixes the correspondence between time-orientable Lorentzian structures $\{g_i\}$ and $H_1^\infty(\mathscr{M}, Z)$. It implies that the properties of FMW-geons considered in the previous section are well defined.

Now let us consider an FMW-geon $\mathscr{W}_k$ related to some time-nonorientable Lorentzian structure g_k, i.e., to a nontrivial tangent line bundle L_k

$$W_1(L_k) = \tau \in H^1(\mathscr{M}, Z_2), \qquad \tau \neq 0 \tag{8}$$

In this case we will assume that the corresponding geon $\mathscr{W}_k$ has to decay. We relate the process of decaying to the existence of time like curves along which the notion of time reverses. However, when the Stiefel–Whitney class is unequal to zero, i.e., $w_1 = \tau \neq 0$, we have no possibility to fix the correspondence between homotopy classes of appropriate line bundles and elements of $H_1^\infty(\mathscr{M}, Z)$ in a natural way. For this reason we will choose only one Lorentzian structure g_τ (see Definition 1) from the whole set determined by a given $\tau \in H^1(\mathscr{M}, Z_2)$. In other words, with any class of isomorphic nontrivial line bundles over $\mathscr{M}$ we will relate only one FMW-geon $\mathscr{W}_\tau$.

Now let us assume that FMW-geons also have the possibility of electromagnetic interactions. It is known that the underlying structure for electromagnetic interactions $\mathscr{F}$ is given by a principal $U(1)$ bundle over a space-time manifold $\mathscr{M}$. Moreover, the set of sectors of interactions $\mathscr{F}$ (or equivalently its vacuums) is given by the subset of flat connections on the $U(1)$ bundle given by $\mathrm{Hom}(\pi_1\mathscr{M}, Z_2) \cong H^1(\mathscr{M}, Z_2)$ (Bugajska, 1987b). We will assume that any FMW-geon $\mathscr{W}_K$ is related to some concrete sector of electromagnetic interactions, i.e., to some concrete element $\tau \in H^1(\mathscr{M}, Z_2)$.

Let us introduce a notation we will use later. Let

$$H_1(\mathscr{M}, Z) = Z^{(1)} \oplus Z^{(2)} \oplus \cdots Z^{(k)} \oplus \cdots \tag{9}$$

Let an element $\tau^k \in \mathrm{Hom}(H_1(\mathscr{M}, Z), Z_2)$ be such that

$$\tau^k(Z^{(i)}) = \begin{cases} 0 & i \neq k \\ Z_2 & i = k \end{cases} \tag{10}$$

Let σ_k denote the generator of the torsionless component $Z^{(k)}$ of $H_1(\mathcal{M}, Z)$. Since $\tau^k(\sigma_k) = -1$ and $\tau^k(\sigma_i) = 0$, $i \neq k$, we have one-to-one correspondence between the generators σ_k of $Z^{(k)}$ and the elements $\tau^k \in \mathrm{Hom}(H_1(\mathcal{M}, Z), Z_2) \cong H^1(\mathcal{M}, Z_2)$ introduced by (10). In other words, to any such element $\tau^k \in H^1(\mathcal{M}, Z_2)$ we can associate a unique element of $H_1(\mathcal{M}, Z)$, namely the generator σ_k of its $Z^{(k)}$ component, and vice versa.

Let us return to the problem of FMW-geons. According to our idea we relate to any geon $\mathcal{W}_i$ some Lorentz structure (i.e., a line bundle L_i) and some element of $H^1(\mathcal{M}, Z_2)$ which determines a "vacuum" of its eletromagnetic interactions. We will assume that any such vacuum sector is represented by elements of the form τ^k described above. Since any element τ^k determines also the isomorphism class of line bundles over $\mathcal{M}$, we see that to any FMW-geon $\mathcal{W}_k$ we can associate a couple

$$(\xi, \eta) \tag{11}$$

of real vector bundles. The first is related to a metric structure of $\mathcal{M}$ and the second to an electromagnetic vaccuum.

Let us recall some facts from fibre bundle theory (Husemoller, 1966). Let ξ and η be two real vector bundles over $\mathcal{M}$. We have the addition function

$$(\xi, \eta) \to \xi \oplus \eta$$

and the multiplication function

$$(\xi, \eta) \to \xi \oplus \eta$$

which introduce a semiring structure in the set S of real vector bundles over $\mathcal{M}$. The ring completion of a semiring S is a pair (S^*, θ) where S^* is a ring and θ: $S \to S^*$ is a morphism which has universal property. This means that for each ring R and for each map f: $S \to R$ there exists a unique map α such that the diagram

$$\begin{tikzcd} & S^* \arrow[d, dashed, "\alpha"] \\ S \arrow[ur, "\theta"] \arrow[r, "f"'] & R \end{tikzcd}$$

is commutative. For the construction of S^* we consider pairs (ξ, η) of $S \times S$. We introduce the following relation:

$$(\xi, \eta) \sim (\xi', \eta') \tag{12}$$

if there exists $\rho \in S$ such that

$$\xi \oplus \eta' \oplus \rho = \xi' \oplus \eta \oplus \rho \tag{12$'$}$$

This is the equivalence relation and $\langle \xi, \eta \rangle$ denotes the equivalence class.

Returning to our physical considerations, i.e., to FMW-geons $\{\mathcal{W}_k\}$, we will [according to our previous considerations and (11)] make the physical distinction between the elements of a pair $(\xi, \eta) \in S \times S$. The first factor will be related to a metric structure of $\mathcal{M}$ and the second to an electromagnetic vacuum.

Moreover, we assume the following convention: if the first factor is a line bundle, then it denotes an appropriate homotopy class of trivial line bundles (we use the bijection $\tau^k \leftrightarrow \sigma_k$). This means that the corresponding Lorentz structure g_k of $\mathcal{M}$ can be connected to the fundamental one g_0 by a Morse function which describes an attaching of a handle of index 1. In this case we say that a geon $\mathcal{W}_k \sim g_k$ has to "realize" or has to "observe" an appropriate surgery of type $(1, 3)$ (or a wormhole). If the Lorentz metric g_i is related to a nontrivial line bundle L_i, then we denote this fact by taking the Whitney sum of L_0 and L_i as the first factor, i.e., in this case we have

$$\xi = L_0 \oplus \tau^i \tag{13}$$

The presence of the bundle L_0 in the first factor ξ means not only that $g_0 \equiv (\tilde{g}, L_0)$ is the exceptional, fundamental Lorentz structure, but also that the Lorentz structure corresponding to ξ possesses a close curve λ related to τ^i [i.e., $\lambda \in \{h^{*-1}(\sigma_i)\}$] along which the notion of time reverses.

In this way to some kind of FMW-geons we can relate a couple of vector bundles. The first bundle ξ can be at most two-dimensional, but the second can be at most linear.

Now let us recall some relations between electromagnetic interactions and spinor fields. Namely, if we had only Majorana spinor fields or (even) two-spinor fields, then we could not introduce the electromagnetic $U(1)$ bundle in a natural way. Only when we take a Dirac spinor bundle does the isomorphism

$$\Sigma \otimes_H \Sigma^* \Leftrightarrow \mathrm{T}\mathcal{M} \tag{14}$$

between the Hermitian part of the tensor product of odd half-spinors Σ and even half-spinors Σ^* and the tangent bundle $T\mathcal{M}$ admit the local $U(1)$ invariance and imply the existence of some $U(1)$ principal bundle (Bugajska, 1985). We regard this bundle as the underlying structure for an electromagnetic gauge. [We also can construct an isomorphism between some part of the tensor product of Majorana spinors and $T\mathcal{M}$, but this map has a different property and does not imply the existence of any $U(1)$ bundle (Bugajska, 1986). This strong relation between Dirac spinors and electromagnetic interactions suggests that particles described by Dirac spinor fields should have some electromagnetic properties even in the case of null electric charge.

Moreover, in the general case, we have a whole set of unequivalent spinor structures on $\mathcal{M}$. This set is given by $H^1(\mathcal{M}, Z_2)$. Each of these "exotic" spinor structures introduces its own $U(1)$ bundle, due to an isomorphism analogous to (14).

However, since all these bundles are isomorphic, we distinguish them by considering different connections on the one $U(1)$ bundle introduced by the ordinary spinor structure (Avis and Isham, 1979). This procedure is equivalent to introducing different flat connections on our $U(1)$ bundle i.e. different vacuums (or sectors) of electromagnetic interactions. In this way we see once again that the spin properties and the electromagnetic properties are strongly related to each other. This seems to be physically justified by the experimental fact that spin can be detected only by means of electromagnetic interactions. However, since we have such an interdependence in the physics of elementary particles, it is natural to assume that we meet a similar situation in the case of FMW-geons. Thus, because we have prescribed some electromagnetic properties to FMW-geons, they also should possess some spin properties.

Let us suppose that there exist geons for which the metric relations of $\mathcal{M}$ are exactly the same as for observers and which do not have any electromagnetic properties. This means that in this case we have the L_0 bundle in the first place and the zero bundle in the second. Hence, to this piece of FMW-geons corresponds the element $\langle L_0, 0\rangle$ in the ring S^*. The fact that we have the couple $(L_0, 0) \in S \times S$ means that we can relate such geons only to two spinors or Majorana spinors. Let us denote this geon by

$$\text{“}\nu\text{”} \sim (L_0, 0) \tag{15}$$

Because "ν" has no electromagnetic properties at all, its electric charge has to be zero. So "ν" geons are the least complicated and the most natural geons in our description. However, besides "ν" there should exist geons that are also related to the fundamental metric structure g_0 but have some nontrivial electromagnetic properties. Thus, for these geons the electromagnetic vacuum should be important, i.e., we should have a nonzero element in the second place of a pair (ξ, η). Since in the ring S^* we have $\langle L_0, L_0\rangle = 0$ as the bundle η, we can only take a line bundle that has the nontrivial Stiefel-Whitney class. Let us assume that this bundle is given by some element $\tau^k \in H^1(M, Z_2)$ introduced in Section 2. So, if we accept the interpretation of FMW-geons described by (b), then we can introduce the same notation for all FMW-geons related to couples of the form (L_0, τ^k). Let us denote them by

$$\text{“}e\text{”} \sim (L_0, \tau^k) \tag{16}$$

Now let us consider FMW-geons related to a metric structure of $\mathcal{M}$ different from g_0. At the beginning let us take geons related to couples of the form (τ^k, L_0). According to our convention we can say that:

(i) A metric structure related to these geons is determined by a trivial line bundle which belongs to the homotopy class of tangent line bundles associated to $\sigma_k \in H_1(\mathcal{M}, Z) \hookrightarrow H_1^\infty(\mathcal{M}, Z)$ (by the correspondence $\tau^k \Leftrightarrow \sigma_k$).

(ii) Since $W_1(L_0) = 0$, the electromagnetic vacuum sector is determined by the canonical flat connection on the principal $U(1)$ electromagnetic bundle.

The presence of τ^k in the first position means that this kind of geon is related to a surgery of type $(1, 3)$. This means nothing more than the "observation" of a "wormhole" between some surfaces ∂U_i and ∂U_{i+1} in a decomposition of $\mathcal{M}$. The homotopy class of a loop related to this surgery (or equivalently to attaching a handle of index 1 to U_i) corresponds, by the natural isomorphism h^*, to the generator σ^k of $H_1(\mathcal{M}, Z)$ or to the element $\tau^k \in H^1(\mathcal{M}, Z_2)$. Similarly as above, we will introduce the same notation for all FMW-geons related to couples of the form (τ, L_0). We will call them "p",

$$\text{“}p\text{”} \sim (\tau^k, L_0) \tag{17}$$

As we have said, the presence of $\eta \neq 0$ in the second position means that these geons have some electromagnetic properties. So, relations between spin and interactions $\mathcal{F}$ imply that such geons are related to Dirac 4-spinors.

We can try to prescribe an electric charge for FMW-geons. Namely, the following situation seems to be the most natural. Since "ν" has no electric charge, all geons related to a pair of vector bundles (ξ, η) that belongs to the same equivalence class as $(L_0, 0)$ (i.e., elements of $S \times S$ that form $\langle L_0, 0\rangle \in S^*$) have no electric charge also. Further, since all FMW-geons related to a couple $(\xi, \eta) = (L_0, \tau^k)$ are different concrete objects of the same quality "e", we prescribe for them the same electric charge, say q. Again, all geons related to a pair of vector bundles (ξ, η) that belongs to $\langle L_0, \tau^k\rangle$ have the same electric charge as "e". Now we can use the following relation valid in the ring S^*:

$$\langle L_0, \tau^k\rangle = -\langle \tau^k, L_0\rangle \tag{18}$$

This relation suggests that according to our convention geon "p" should have an electric charge opposite to the electric charge of geon "e", i.e., $-q$.

Now let us consider FMW-geons associated to Lorentz structures $g_i \equiv (\tilde{g}, L_i)$ with $W_1(L_i) \neq 0$. Let us consider the case when $W_1(L_i) = \tau^i$. In this case the first factor in the pair (ξ, η) of vector bundles has to be equal to $\xi = L_0 \oplus \tau^i$. Let us consider the couple

$$(L_0 \oplus \tau^k, \tau^k) \tag{19}$$

According to our convention, these geons are unstable, since they are associated to a loop along which the notion of time reverses [the homotopy class of this loop determines $\sigma^k \in H_1(\mathcal{M}, Z)$ or equivalently $\tau^k \in H^1(M, Z_2)$

as mentioned above]. Moreover, although electromagnetic properties of a geon related to a couple (19) do not vanish (i.e., these geons are related to Dirac 4-spinors), the electric charge of this geon has to be equal to zero ($\langle L_0, 0\rangle = \langle L_0 \oplus \tau^k, \tau^k\rangle$ in S^*). We will denote these FMW-geons by "m", i.e.,

$$\text{“}m\text{”} \sim (L_0 \oplus \tau^k, \tau^k) \tag{20}$$

As a matter of fact we have no other possibility to fix the second factor in (19). Namely, according to our convention and according to the rules in the ring S^* an electric charge equal to the electric charge of geon "e" can be represented by elements of the form $(L_0 \oplus A, A \oplus \tau^k)$ with nontrivial A. Also, an electric charge equal to the charge of geon "p" can be given by $(\tau^k + B, B + L_0)$ with nontrivial B. However, our interpretation of the second factor as an electromagnetic vacuum excludes both these possibilities. It excludes also the possibility of the existence of FMW-geons associated to a couple of the form $(\tau^k, 0)$. Namely, such a pair of real vector bundles determines another (other than "e" or "p") electric charge, but simultaneously a geon related to it cannot have any electromagnetic properties since $\eta = 0$.

In this way when we consider gravitational and electromagnetic properties of FMW-geons we can introduce only four qualitatively different types of FMW-geons. We have denoted them by "ν," "e," "p," and "m." The first three are stable, whereas "m" has to decay. Now in the ring S^* we have the following rules:

$$\langle L_0 \oplus \tau^k, \tau^k\rangle = \langle L_0, \tau^k\rangle + \langle \tau^k, 0\rangle \tag{21}$$

$$= \langle L_0, \tau^k\rangle + \langle \tau^k, L_0\rangle + \langle L_0, 0\rangle \tag{21'}$$

Since an element $\langle \tau^k, p\rangle$ cannot be related to any FMW-geon, we should consider only the formula (21′). We can rewrite it as

$$\text{“}m\text{”} \rightarrow \text{“}\nu\text{”} + \text{“}p\text{”} + \text{“}e\text{”}$$

Although there exist other possibilities than (21) of the "decomposition" of "m" in the ring S^*, it is easy to check that they also are excluded by our convention.

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